Semi-parametric Pricing and Hedging of Volatility and Hybrid Derivatives

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Pricing of exotic derivatives: parametric approach

The **parametric approach** for valuing exotic derivatives involves:

- specifying parametric risk-neutral dynamics for the spot price S of an underlying asset (e.g., Black Scholes, CEV, Heston, SABR, Hull-White, MJD, Variance Gamma, CGMY, ...),
- calibrating the parameters to liquid market prices of puts $P(K_i, T_j)$ & calls $C(K_i, T_j)$, typically by a least squares fit,
- valuing exotics (either analytically or numerically) under the specification using the calibrated parameters.

This approach has a number of shortcomings:

- parametric models rarely exactly fit market data,
- the parameters can be difficult to identify and the calibration procedure is often computationally intensive. As time evolves, the fit worsens, requiring re-calibration,
- to speed up the (re-)calibration, the dynamical specification is often chosen to produce closed-form call/put prices, but these are rare and possibly far from market prices.

Pricing of exotic derivatives: non-parametric approach

The **non-parametric approach** to pricing exotics involves:

- specifying non-parametric risk-neutral dynamics for the underlying spot price S (e.g., under zero rates & dividends, S is a driftless diffusion or a positive continuous martingale),
- converting discrete strike/maturity option prices into arbitrage-free curves $P/C(T)$, $P/C(K)$, or surfaces $P/C(K,T)$,
- deriving **upper and/or lower bounds** for exotic derivative prices consistent with the curve or surface. Sometimes, the bounds meet, eg. for (continuously monitored) variance swaps relative to $P/C(K)$.

This approach has some possible shortcomings:

- difficult to interpolate/extrapolate $P/C(K_i, T_j)$ arbitrage-free.
- sub and super-replication strategies typically rule out dynamic trading in options ex ante; thereby widening the no arbitrage bounds. The resulting lower and upper bounds may be too far apart to use as the bid and ask price of the exotic.

Pricing of exotic derivatives: semi-parametric approach

The **semi-parametric approach** we use can be outlined as follows:

- specify part of the risk-neutral dynamics of S parametrically, with the rest specified non-parametrically,
- when the exotic's payoff depends on $[\ln S]_T$ and possibly also S_T, we get **unique** prices and hedges **relative** to given co-terminal European call prices $C(K)$ and put prices $P(K)$.

This approach has a number of advantages:

- **compared to parametric models**, semi-parametric models are more flexible and therefore more likely to fit market data,
- **compared to the typical usage of non-parametric models**, our replicating strategy allows dynamic trading in calls and puts, causing the upper and lower bounds on value to meet.

Basic assumptions and notation

Throughout this talk, we make the following assumptions:

- no arbitrage,
- no transactions costs,
- zero interest rates/dividends.

We fix a maturity date T.

Denote by $S = (S_t)_{0 \leq t \leq T}$ the price of a strictly positive risky asset.

Denote by $X = (X_t)_{0 \leq t \leq T}$ the $\ln$ price: $X_t = \ln S_t$.

Under the above assumptions, put and call prices are given by

$$P(K) = \mathbb{E}(K - S_T)^+, \qquad C(K) = \mathbb{E}(S_T - K)^+.$$

Here, $\mathbb{E}$ denotes expectation with respect to the market's chosen risk-neutral pricing measure $\mathbb{Q}$.

We assume a call and/or put trades at every strike $K \in (0, \infty)$.

Non-parametric pricing of Path-Independent Payoffs

Carr and Madan (1998) show that, if f can be expressed as the difference of convex functions, then for any $\kappa \in \mathbb{R}^+$, we have

$$
\begin{aligned}
f(s) &= f(\kappa) + f'(\kappa)\Big((s-\kappa)^+ - (\kappa - s)^+\Big) \\
&\quad + \int_0^\kappa \mathrm{d}K\, f''(K)(K-s)^+ + \int_\kappa^\infty \mathrm{d}K\, f''(K)(s-K)^+.
\end{aligned}
$$

Replacing s with S_T, setting $\kappa = S_0$, and taking an expectation

$$
\mathbb{E}\, f(S_T) = f(S_0) + \int_0^{S_0} \mathrm{d}K\, f''(K) P(K) + \int_{S_0}^\infty \mathrm{d}K\, f''(K) C(K).
$$

Takeaway: the price of any path-independent payoff $\mathbb{E} f(S_T)$ can be expressed relative to market prices of puts and calls on S_T.

This result makes no assumptions on the dynamics of the spot price process S.

To price path-dependent payoffs, we need to impose some structure on the risk-neutral dynamics of S.

Our Semi-parametric model

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ the underlying asset's spot price S solves:

$$\mathrm{d}S_t = \sigma_t S_t \mathrm{d}W_t + \int_{\mathbb{R}} (\mathrm{e}^z - 1) S_{t-} \widetilde{N}(\mathrm{d}t, \mathrm{d}z),$$

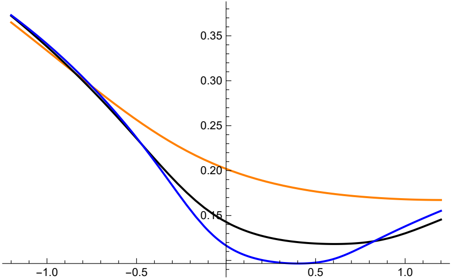
$$\widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t,$$

- W is a Brownian motion under the risk-neutral pricing measure $\mathbb{Q}$, with respect to (w.r.t.) the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$.
- $\widetilde{N}$ is a compensated Poisson random measure w.r.t. $\mathbb{Q}$.
- The volatility process σ evolves independently of S,W, and $\widetilde{N}$.

The model is **semi-parametric** in that:

- The vol process σ is non-parametric (σ need not be Markov, eg. fractional Brownian motion w. unknown Hurst parameter, and may jump with unknown intensity/jump size).
- We specify the risk-neutral Lévy measure ν parametrically.

Framework allows for asymmetric implied volatility smiles



Imp. vol as a function of ln-moneyness-to-maturity for $T = \{1, 2, 3\}$ months.

$$\mathrm{d}X_t = \gamma(Z_t)\mathrm{d}t + \sqrt{Z_t}\mathrm{d}W_t + \int_{\mathbb{R}} z\widetilde{N}(\mathrm{d}t, \mathrm{d}z),$$

$$\mathrm{d}Z_t = \kappa(\theta - Z_t)\mathrm{d}t + \delta\sqrt{Z_t}\mathrm{d}B_t,$$

$$\nu(\mathrm{d}z) = \frac{1}{\sqrt{2\pi s^2}} \exp\left(\frac{-(z-m)^2}{2s^2}\right)\mathrm{d}z.$$

Types of claims we consider

By Itô's Lemma, the process $X := \ln S$ satisfies

$$\begin{aligned}\mathrm{d}X_t &= -\tfrac{1}{2}\sigma_t^2\mathrm{d}t + \sigma_t\mathrm{d}W_t \\ &\quad - \int_{\mathbb{R}}(\mathrm{e}^z - 1 - z)\nu(\mathrm{d}z)\mathrm{d}t + \int_{\mathbb{R}} z\widetilde{N}(\mathrm{d}t, \mathrm{d}z).\end{aligned}$$

We wish to price and hedge **hybrid claims** of the form

$$\text{Payoff at time } T = \varphi(X_T, [X]_T),$$

$$[X]_T = \text{realized quadratic variation of } X \text{ up to time } T.$$

Examples

Variance Swap : $\varphi(X_T, [X]_T) = [X]_T,$

Volatility Swap : $\varphi(X_T, [X]_T) = \sqrt{[X]_T},$

Sharpe Ratio : $\varphi(X_T, [X]_T) = (X_T - X_0)/\sqrt{[X]_T}.$

We also consider options on Leveraged ETFs, which are path-dependent claims on X, but whose payoff cannot be written simply as $\varphi(X_T, [X]_T)$.

Pricing exponential claims

We use exponential claims as a basis for more general claims.

$$\text{exponential claim payoff :} \qquad e^{i\omega X_T + is[X]_T}$$

To this end, the following proposition will be useful.

Proposition

Define $u : \mathbb{C}^2 \to \mathbb{C}$ *and* $\psi : \mathbb{C}^2 \to \mathbb{C}$ *as*

$$u(\omega, s) := i\left(-\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \omega^2 - i\omega + 2is}\right),$$

$$\psi(\omega, s) := \int_{\mathbb{R}} \nu(dz)\left(e^{i\omega z + isz^2} - 1 - i\omega(e^z - 1)\right).$$

Then the joint characteristic function of $(X_T, [X]_T)$ *given* $\mathcal{F}_t$ *is*

$$\underbrace{\mathbb{E}_t e^{i\omega X_T + is[X]_T}}_{\textit{Path-dep. claim}} = \underbrace{\frac{e^{(T-t)\psi(\omega,s) + i(\omega - u(\omega,s))X_t + is[X]_t}}{e^{(T-t)\psi(u(\omega,s),0)}}}_{\mathcal{F}_t\textit{-measurable}} \underbrace{\mathbb{E}_t e^{iu(\omega,s)X_T}}_{\textit{Path-ind. claim}}.$$

Key ingredients of proof

X can be separated into a continuous component and an independent jump component

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}X_t^c + \mathrm{d}X_t^j, \\ \mathrm{d}X_t^c &= -\tfrac{1}{2}\sigma_t^2 \mathrm{d}t + \sigma_t W_t, \\ \mathrm{d}X_t^j &= -\int_{\mathbb{R}} (\mathrm{e}^z - 1 - z)\nu(\mathrm{d}z)\mathrm{d}t + \int_{\mathbb{R}} z\widetilde{N}(\mathrm{d}t, \mathrm{d}z). \end{aligned}$$

Carr and Lee (2008) show that the continuous component $(X^c, [X^c])$ satisfies

$$\mathbb{E}_t \mathrm{e}^{\mathtt{i}\omega(X_T^c - X_t^c) + \mathtt{i}s([X^c]_T - [X^c]_t)} = \mathbb{E}_t \mathrm{e}^{\mathtt{i}u(\omega,s)(X_T^c - X_t^c)}.$$

The jump component $(X^j, [X^j])$ is a two-dimensional Lévy process with joint characteristic exponent ψ

$$\mathbb{E}_t \mathrm{e}^{\mathtt{i}\omega(X_T^j - X_t^j) + \mathtt{i}s([X^j]_T - [X^j]_t)} = \mathrm{e}^{(T-t)\psi(\omega,s)},$$

Proof

Using results from the previous page, we have

$$
\begin{aligned}
&\mathbb{E}_t e^{\mathrm{i}\omega(X_T - X_t) + \mathrm{i}s([X]_T - [X]_t)} \\
&= \mathbb{E}_t e^{\mathrm{i}\omega(X^c_T - X^c_t) + \mathrm{i}s([X^c]_T - [X^c]_t)} \\
&\quad \mathbb{E}_t e^{\mathrm{i}\omega(X^j_T - X^j_t) + \mathrm{i}s([X^j]_T - [X^j]_t)} && (X^c \perp\!\!\!\perp X^j) \\
&= \mathbb{E}_t e^{\mathrm{i}u(\omega,s)(X^c_T - X^c_t)} && \text{(Carr Lee result)} \\
&\quad e^{(T-t)\psi(\omega,s)} && ((X^j, [X^j]) \text{ is Lévy}) \\
&= \frac{\mathbb{E}_t e^{\mathrm{i}u(\omega,s)(X_T - X_t)}}{\mathbb{E}_t e^{\mathrm{i}u(\omega,s)(X^j_T - X^j_t)}} e^{(T-t)\psi(\omega,s)} && (X^c \perp\!\!\!\perp X^j) \\
&= \frac{\mathbb{E}_t e^{\mathrm{i}u(\omega,s)(X_T - X_t)}}{e^{(T-t)\psi(u(\omega,s),0)}} e^{(T-t)\psi(\omega,s)}. && (X^j \text{ is Lévy})
\end{aligned}
$$

Thus, we obtain

$$
\mathbb{E}_t e^{\mathrm{i}\omega X_T + \mathrm{i}s[X]_T} = \frac{e^{-\mathrm{i}u(\omega,s)X_t} e^{\mathrm{i}\omega X_t + \mathrm{i}s[X]_t}}{e^{(T-t)\psi(u(\omega,s),0)}} e^{(T-t)\psi(\omega,s)} \mathbb{E}_t e^{\mathrm{i}u(\omega,s)X_T}.
$$

Pricing power-exponential claims

We can use previous result to price power-exponential claims

$$\underbrace{\mathbb{E}_t X_T^n [X]_T^m \mathrm{e}^{\mathrm{i}\omega X_T + \mathrm{i}s[X]_T}}_{\text{power-exponential claim price}}$$

$$= (-\mathrm{i}\partial_\omega)^n (-\mathrm{i}\partial_s)^m \mathbb{E}_t \mathrm{e}^{\mathrm{i}\omega X_T + \mathrm{i}s[X]_T}$$

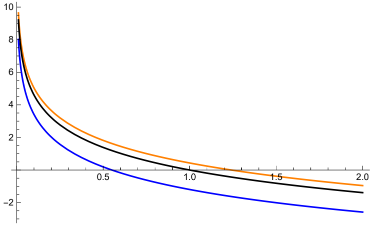
$$= (-\mathrm{i}\partial_\omega)^n (-\mathrm{i}\partial_s)^m \underbrace{\frac{\mathrm{e}^{(T-t)\psi(\omega,s) + \mathrm{i}(\omega - u(\omega,s))X_t + \mathrm{i}s[X]_t}}{\mathrm{e}^{(T-t)\psi(u(\omega,s),0)}}}_{=:F(\omega,s,X_t,[X]_t)} \mathbb{E}_t \mathrm{e}^{\mathrm{i}u(\omega,s)X_T}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{m} \underbrace{\binom{n}{j}\binom{m}{k} (-\mathrm{i}\partial_\omega)^j (-\mathrm{i}\partial_s)^k F(\omega, s, X_t, [X]_t)}_{\mathcal{F}_t\text{-measurable}}$$

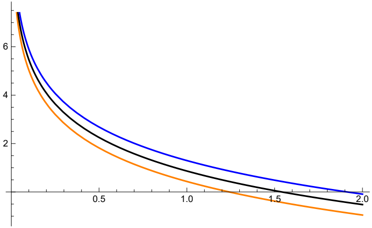
$$\times \underbrace{\mathbb{E}_t (-\mathrm{i}\partial_\omega)^{n-j} (-\mathrm{i}\partial_s)^{m-k} \mathrm{e}^{\mathrm{i}u(\omega,s)X_T}}_{\text{Path-independent claim price}}$$

Example: variance swap

Effect of jump size



Effect of jump intensity



We plot $g(\ln s)$ as a function of s where

$$\mathbb{E}g(\ln S_T) = \mathbb{E}[\ln S]_T, \qquad \nu(\mathrm{d}z) = \lambda\delta_m(z)\mathrm{d}z,$$

$$T = 0.25, \qquad S_0 = 1.$$

Left : $\lambda = 1.00, \qquad m = \{-2.00, 0, 2.00\},$

Right : $m = -2.00, \qquad \lambda = \{1.00, 2.00, 3.00\}.$

Pricing fractional powers of $[X]_T$

We use the following integral representation

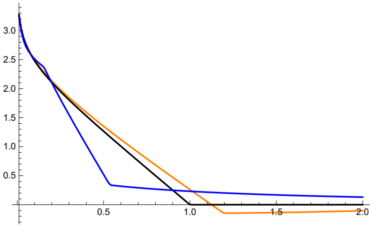
$$v^r = \frac{r}{\Gamma(1-r)} \int_0^\infty \mathrm{d}z \, \frac{1}{z^{r+1}} \left(1 - \mathrm{e}^{-zv}\right), \qquad 0 < r < 1,$$

where Γ is the Gamma function. Setting $X_0 = 0$, we have

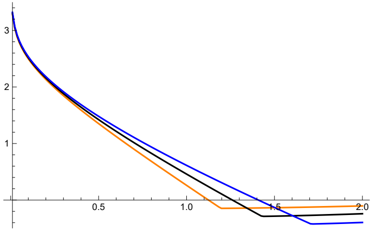
$$\begin{aligned}
&\frac{\Gamma(1-r)}{r} \mathbb{E}[X]_T^r \\
&= \int_0^\infty \mathrm{d}z \, \frac{1}{z^{r+1}} \Big(\underbrace{\mathbb{E}1 - \mathbb{E}\mathrm{e}^{-z[X]_T}}_{\text{exponential claims}} \Big) \\
&= \mathbb{E} \int_0^\infty \mathrm{d}z \, \frac{1}{z^{r+1}} \left(\mathrm{e}^{\mathtt{i}u(0,0)X_T} - \frac{\mathrm{e}^{T\psi(0,\mathtt{i}z)}}{\mathrm{e}^{T\psi(u(0,\mathtt{i}z),0)}} \mathrm{e}^{\mathtt{i}u(0,\mathtt{i}z)X_T} \right) \\
&=: \frac{\Gamma(1-r)}{r} \mathbb{E}g(X_T).
\end{aligned}$$

Example: volatility swap

Effect of jump size



Effect of jump intensity



We plot $g(\ln s)$ as a function of s where

$$\mathbb{E}g(\ln S_T) = \mathbb{E}\sqrt{[\ln S]_T}, \quad \nu(\mathrm{d}z) = \lambda\delta_m(z)\mathrm{d}z, \quad T = 0.25.$$

Left : $\lambda = 1.00,$ $m = \{-1.25, 0.00, 1.25\},$

Right : $m = -1.25,$ $\lambda = \{1.00, 2.00, 3.00\}.$

Pricing ratio claims $X_T^n/([X]_T+\varepsilon)^r$

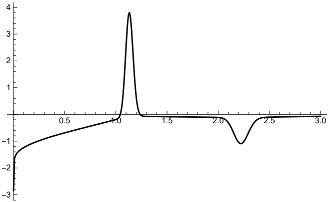
Using the integral representation

$$\frac{1}{(v+\varepsilon)^r} = \frac{1}{r\Gamma(r)}\int_0^\infty \mathrm{d}z\, \mathrm{e}^{-z^{1/r}(v+\varepsilon)}, \qquad r>0,$$

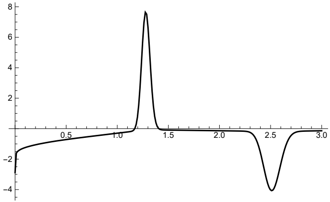
we have

$$\begin{aligned}
&\mathbb{E}\frac{X_T^n \mathrm{e}^{\mathtt{i}pX_T}}{([X]_T+\varepsilon)^r}\\
&= \frac{1}{r\Gamma(r)}\int_0^\infty \mathrm{d}z\, \mathbb{E}X_T^n \mathrm{e}^{\mathtt{i}pX_T - z^{1/r}([X]_T+\varepsilon)}\\
&= \frac{1}{r\Gamma(r)}\int_0^\infty \mathrm{d}z\, \mathrm{e}^{-z^{1/r}\varepsilon}(-\mathtt{i}\partial_p)^n \underbrace{\mathbb{E}\mathrm{e}^{\mathtt{i}pX_T - z^{1/r}[X]_T}}_{\text{exponential claim price}}\\
&= \frac{1}{r\Gamma(r)}\mathbb{E}\int_0^\infty \mathrm{d}z\, \mathrm{e}^{-z^{1/r}\varepsilon}(-\mathtt{i}\partial_p)^n \frac{\mathrm{e}^{T\psi(p,\mathtt{i}z^{1/r})}}{\mathrm{e}^{T\psi(u(p,\mathtt{i}z^{1/r}),0)}}\mathrm{e}^{\mathtt{i}u(p,\mathtt{i}z^{1/r})X_T}\\
&=: \mathbb{E}g(X_T).
\end{aligned}$$

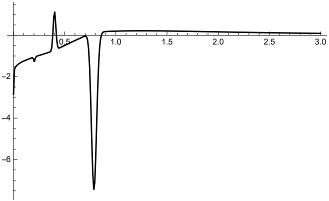
Example: realized Sharpe ratio



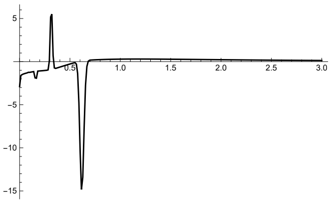
$\lambda = 1.0$, $m = -0.675$



$\lambda = 2.0$, $m = -0.675$



$\lambda = 1.0$, $m = 0.675$



$\lambda = 2.0$, $m = 0.675$

We plot $g(\ln s)$ as a function of s where $\varepsilon = 0.001$ and

$$\mathbb{E}g(\ln S_T) = \mathbb{E}X_T / \sqrt{[\ln S]_T + \varepsilon}, \qquad \nu(\mathrm{d}z) = \lambda \delta_m(z)\mathrm{d}z.$$

Leveraged ETFs

The relationship between an Leveraged Exchange Traded Fund (LETF) $L = \mathrm{e}^{Y}$ and the underlying Exchange Traded Fund ETF $S = \mathrm{e}^{X}$ is

$$\frac{\mathrm{d}L_t}{L_{t-}} = \beta \frac{\mathrm{d}S_t}{S_{t-}},$$

where $\beta \in \{-2, -1, 2, 3\}$ is the **leverage ratio**.

Here, we assume a jump in S will not send L to a negative value.

The value of Y_T depends on the **path** of X as follows

$$\begin{aligned}
\mathrm{d}Y_t &= \mathrm{d}Y_t^c + \mathrm{d}Y_t^j, \\
\mathrm{d}Y_t^c &= \beta \mathrm{d}X_t^c + \tfrac{1}{2}\beta(1-\beta)\mathrm{d}[X^c]_t, \\
\mathrm{d}Y_t^j &= -\int_{\mathbb{R}} \Big(\beta(\mathrm{e}^z - 1) - \ln\big(\beta(\mathrm{e}^z - 1) + 1\big)\Big)\nu(\mathrm{d}z)\mathrm{d}t \\
&\quad + \int_{\mathbb{R}} \ln\big(\beta(\mathrm{e}^z - 1) + 1\big)\widetilde{N}(\mathrm{d}t, \mathrm{d}z).
\end{aligned}$$

Characteristic Function of Y_T

While Y_T depends on the path of X, we can relate the characteristic function of Y_T to the characteristic function of X_T only:

Proposition

Define $\chi : \mathbb{C} \to \mathbb{C}$ *by*

$$\chi(q) := \int_{\mathbb{R}} \nu(\mathrm{d}z)\Big(\big(\beta(\mathrm{e}^z - 1) + 1\big)^{\mathtt{i}q} - 1 - \mathtt{i}q\beta(\mathrm{e}^z - 1)\Big).$$

Then the characteristic function of $(Y_T - Y_t)$*, conditional on* $\mathcal{F}_t$*, is*

$$\underbrace{\mathbb{E}_t \mathrm{e}^{\mathtt{i}q(Y_T - Y_t)}}_{\textit{Path-dep. claim}} = \underbrace{\frac{\mathrm{e}^{(T-t)\chi(q)}}{\mathrm{e}^{(T-t)\psi(u(q\beta, q\frac{1}{2}\beta(1-\beta)),0)}}}_{\mathcal{F}_t\textit{-measurable}} \underbrace{\mathbb{E}_t \mathrm{e}^{\mathtt{i}u(q\beta, q\frac{1}{2}\beta(1-\beta))(X_T - X_t)}}_{\textit{Path-independent claim}},$$

where u *and* ψ *as defined previously.*

Proof

Using

$$\mathbb{E}_t \mathrm{e}^{\mathtt{i} q (Y_T^j - Y_t^j)} = \mathrm{e}^{(T-t)\chi(q)}, \qquad (1)$$

$$\mathbb{E}_t \mathrm{e}^{\mathtt{i}\omega (X_T^j - X_t^j) + \mathtt{i} s ([X^j]_T - [X^j]_t)} = \mathrm{e}^{(T-t)\psi(\omega, s)}, \qquad (2)$$

and independence of continuous and jump components, we have

$$\mathbb{E}_t \mathrm{e}^{\mathtt{i} q (Y_T - Y_t)} = \mathbb{E}_t \mathrm{e}^{\mathtt{i} q (Y_T^c - Y_t^c)} \mathbb{E}_t \mathrm{e}^{\mathtt{i} q (Y_T^j - Y_t^j)} \qquad (Y^c \perp\!\!\!\perp Y^j)$$

$$= \mathbb{E}_t \mathrm{e}^{\mathtt{i} q \beta (X_T^c - X_t^c) + \mathtt{i} q \frac{1}{2} \beta (1-\beta)([X^c]_T - [X^c]_t)} \mathrm{e}^{(T-t)\chi(q)} \qquad \text{(by (1))}$$

$$= \mathbb{E}_t \mathrm{e}^{\mathtt{i} u (q\beta, q\frac{1}{2}\beta(1-\beta))(X_T^c - X_t^c)} \mathrm{e}^{(T-t)\chi(q)} \qquad \text{(Carr Lee result)}$$

$$= \frac{\mathbb{E}_t \mathrm{e}^{\mathtt{i} u (q\beta, q\frac{1}{2}\beta(1-\beta))(X_T - X_t)}}{\mathbb{E}_t \mathrm{e}^{\mathtt{i} u (q\beta, q\frac{1}{2}\beta(1-\beta))(X_T^j - X_t^j)}} \mathrm{e}^{(T-t)\chi(q)} \qquad (X^c \perp\!\!\!\perp X^j)$$

$$= \frac{\mathbb{E}_t \mathrm{e}^{\mathtt{i} u (q\beta, q\frac{1}{2}\beta(1-\beta))(X_T - X_t)}}{\mathrm{e}^{(T-t)\psi(u(q\beta, q\frac{1}{2}\beta(1-\beta)), 0)}} \mathrm{e}^{(T-t)\chi(q)}. \qquad \text{(by (2))}$$

Pricing general claims on Y_T

Let $\widehat{\varphi}$ be the (possibly generalized) Fourier transform of φ

$$\textbf{Fourier Transform}: \qquad \widehat{\varphi}(q) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}y\, \mathrm{e}^{-\mathtt{i}qy}\varphi(y),$$

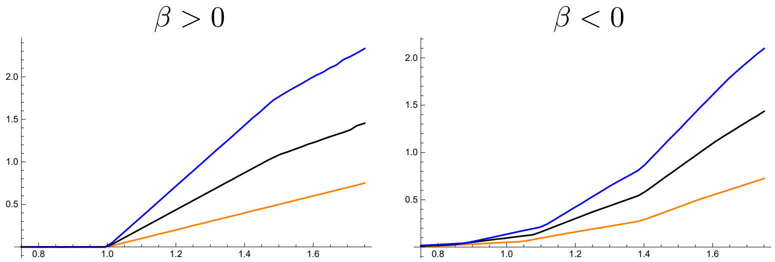
$$\textbf{Inverse Transform}: \qquad \varphi(y) = \int_{\mathbb{R}} \mathrm{d}q_r\, \mathrm{e}^{\mathtt{i}qy}\widehat{\varphi}(q),$$

where q_r is the *real part* of q.

The price of a claim with payoff $\varphi(Y_T)$ can be obtained as follows

$$\begin{aligned}
&\mathbb{E}_t\varphi(Y_T) \\
&= \int_{\mathbb{R}} \mathrm{d}q_r \widehat{\varphi}(q)\mathrm{e}^{\mathtt{i}qY_t}\mathbb{E}_t\mathrm{e}^{\mathtt{i}q(Y_T-Y_t)} \\
&= \int_{\mathbb{R}} \mathrm{d}q_r \widehat{\varphi}(q)\mathrm{e}^{\mathtt{i}qY_t}\frac{\mathrm{e}^{(T-t)\chi(q)}}{\mathrm{e}^{(T-t)\psi(u(q\beta,q\frac{1}{2}\beta(1-\beta)),0)}}\mathbb{E}_t\mathrm{e}^{\mathtt{i}u(q\beta,q\frac{1}{2}\beta(1-\beta))(X_T-X_t)} \\
&=: \underbrace{\mathbb{E}_t g(X_T;X_t,Y_t)}_{\text{Path-indep. claim price}} .
\end{aligned}$$

Example: Calls on L_T



We plot $g(\ln s; x, y)$ as a function of s where

$$\mathbb{E}g(\ln S_T; X_0, Y_0) = \mathbb{E}(L_T - K)^+, \qquad \nu(\mathrm{d}z) = \lambda\delta_m(z)\mathrm{d}z.$$

where $K = 1.0$, $T = 1/4$, $X_0 = Y_0 = 0.0$, $m = -0.4$ and $\lambda = 2.0$.

Left : $\beta = \{1.0, 2.0, 3.0\}$, Right : $\beta = \{-1.0, -2.0, -3.0\}$.

Replicating Exponential Claims

The value of an exponential claim at any time $t \leq T$ is

$$\mathbb{E}_t e^{i\omega X_T + is[X]_T} = A_t Q_t^{(u)}, \qquad u \equiv u(\omega, s),$$

where we have defined

$$A_t := \underbrace{e^{i(\omega-u)X_t + is[X]_t} \frac{e^{(T-t)\psi(\omega,s)}}{e^{(T-t)\psi(u,0)}}}_{\mathcal{F}_t\text{-measurable}}, \qquad Q_t^{(u)} := \underbrace{\mathbb{E}_t e^{iuX_T}}_{\text{Path-ind. claim}}.$$

To derive the hedging strategy, take the differential

$$d(A_t Q_t^{(u)}) = A_t dQ_t^{(u)} + Q_{t-}^{(u)} dA_t + d[A, Q^{(u)}]_t,$$

and show that the right-hand side can be expressed as a **self-financing portfolio** of **traded assets**, namely

- the underlying stock S
- zero-coupon bonds B
- European exponential claims $Q^{(q)}$ where $q \in \mathbb{C}$.

Key Ingredients in the Derivation

- Jump in value of European claim $Q_t^{(q)} = \mathbb{E}_t \mathrm{e}^{\mathtt{i} q X_T}$ is

$$\Delta Q_t^{(q)} = Q_{t-}\left(\mathrm{e}^{\mathtt{i} q \Delta X_t} - 1\right) + \text{jump due to } \Delta\sigma_t.$$

- We also have the following **symmetry**

$$R_t^{(q)} Q_t^{(q)} = R_t^{(-\mathtt{i}-q)} Q_t^{(-\mathtt{i}-q)},$$

where the process $R^{(q)}$ is given by

$$R_t^{(q)} = \mathrm{e}^{-\mathtt{i} q X_t + (T-t)\psi(-\mathtt{i}-q,0)}.$$

Explicit Replication Strategy

We have (traded assets in blue)

$$
\begin{aligned}
\mathrm{d}(A_t Q_t^{(u)}) = A_{t-}\mathrm{d}Q_t^{(u)} &+ \mathtt{i}(\omega - u)\frac{A_{t-}Q_{t-}^{(u)}}{S_{t-}}\mathrm{d}S_t \\
&+ \sum_{j=1}^m H_{t-}^{(j)} \left(R_{t-}^{(q_j)}\mathrm{d}Q_t^{(q_j)} - R_{t-}^{(-\mathtt{i}-q_j)}\mathrm{d}Q_t^{(-\mathtt{i}-q_j)} \right) \\
&+ \sum_{j=1}^m H_{t-}^{(j)}(1 - 2\mathtt{i}q_j)\frac{R_{t-}^{(q_j)}Q_{t-}^{(q_j)}}{S_{t-}}\mathrm{d}S_t,
\end{aligned}
$$

where $H \in \mathbb{C}^m$ satisfies

$$
0 = \Delta\Gamma_t^{(u)} + \sum_{j=1}^m H_t^{(j)} \left(\Delta\Omega_t^{(q_j)} - \Delta\Omega_t^{(-\mathtt{i}-q_j)} \right),
$$

and for any $q \in \mathbb{C}$, the processes $\Gamma^{(u)}$ and $\Omega^{(q)}$ are given by

$$
\mathrm{d}\Gamma_t^{(u)} := A_{t-}Q_{t-}^{(u)} \int_{\mathbb{R}} \left(\mathrm{e}^{\mathtt{i}\omega z + \mathtt{i}sz^2} - \mathrm{e}^{\mathtt{i}uz} - \mathtt{i}(\omega - u)(\mathrm{e}^z - 1) \right) N(\mathrm{d}t, \mathrm{d}z),
$$

$$
\mathrm{d}\Omega_t^{(q)} := R_{t-}^{(q)}Q_{t-}^{(q)} \int_{\mathbb{R}} \left(-\mathrm{e}^{\mathtt{i}qz} + 1 + \mathtt{i}q(\mathrm{e}^z - 1) \right) N(\mathrm{d}t, \mathrm{d}z).
$$

Conclusion

- We specified **semi-parametric dynamics** for a spot price S
  - The volatility process σ is non-parametric; it may be non-Markovian (e.g., driven by fBM) and may jump
  - In contrast. the jumps in log price X are specified parametrically via the risk-neutral Lévy measure ν
  - Asymmetric jumps in X lead to **asymmetric implied volatility** smiles
- We have shown how to price the following path-dependent claims relative to the market prices of European calls and puts
  - claims written purely on "realized variance" (i.e the quadratic variation of log price).
  - hybrid claims on both realized variance and final spot price
  - options on LETFs
- We have shown how to **replicate** exponential claims with a self-financing portfolio of traded assets. Since the family of complex exponential payoffs form a basis, almost any other payoff is also replicable.